# On "Testing group commutativity" by F.Magniez and A.Nayak

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# Introduction

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When do we use black box groups?

# Group Commutativity

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*Task:* Determine whether $G$ is abelian

# Classical algorithms for Group commutativity

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- Randomized algorithm with query complexity $\Theta(k)$ [I.Pak, 2000]. This is optimal *randomized* algorithm up to a constant [F.Magniez, A.Nayak, 2005]

# Randomized algorithm for group commutativity

*Definition.* Define random subproduct as

$$h = g_1^{a_1} \dots g_k^{a_k},$$

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# Quantum algorithm

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- Evaluate the quantities in

$$S + \frac{1}{\sqrt{\delta\varepsilon}}(U + C)$$

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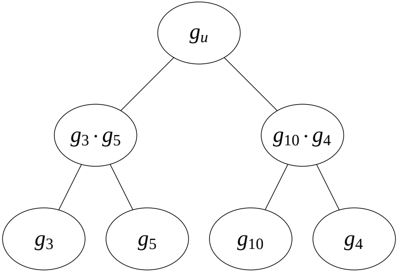
*Example.* Let $l = 4, k = 20, u = \{3, 5, 10, 4\} \in S_4$. Then
$g_u = g_3 \cdot g_5 \cdot g_{10} \cdot g_4$ and $t_u$ looks as follows



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    3. Update tree $t_u$

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Vertex $(t_u, t_v)$ is marked iff $g_u g_v \neq g_v g_u$.

# Evaluating parameters – the fraction of marked vertices

*Lemma.* If $G$ is not abelian and $l = o(k)$ then

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Thus, fraction of marked vertices, $\varepsilon = \Omega\left(\left(\frac{l}{k}\right)^2\right)$

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Thus, the spectral gap, $\delta = \Omega\left(\frac{1}{l \log l}\right)$

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- Checking cost $C = O(1)$

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To minimize quantum query complexity we set $l = k^{2/3}$ and get

$$O(k^{2/3} \log k)$$

# Lower bounds

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### Unique split collision

Output YES if there exists a unique pair $x, y$ such that $x \in \{1, \ldots, \frac{k}{2}\}, y \in \{\frac{k}{2} + 1, \ldots, k\}$ such that $f(x) = f(y)$.

### Theorem

The quantum query complexity of unique split collision is $\Omega(k^{2/3})$.

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The quantum query complexity of unique split collision is $\Omega(k^{2/3})$.

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Idea: Reduce *unique split collision* to group commutativity by constructing a group that is commutative iff function $f$ has a unique split collision.

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- Classical query complexity is $\Theta(k)$.
- Quantum query complexity is upper bounded by $O(k^{2/3} \log k)$ (algorithm based on Q-walk) and lower bounded by $\Omega(k^{2/3})$.